Soliton evolution and radiation loss for the nonlinear Schrödinger equation

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The transient evolution of general initial pulses into solitons for the nonlinear Schrödinger (NLS) equation is considered. By employing a trial function which consists of a solitonlike pulse with variable parameters plus a linear dispersive term in an averaged Lagrangian, ordinary differential equations (ODE's) are derived which approximate this evolution. These approximate equations take into account the effect of the generated dispersive radiation upon the pulse evolution. Specifically, in the approximate ODE's the radiation acts as a damping which causes the pulse to decay to a steady soliton. The solutions of the approximate ODE's are compared with numerical solutions of the NLS equation and are found to be in very good agreement. In addition, the potential implications for obtaining improved approximate ODE models for soliton propagation in optical fibers and other devices governed by NLS-type equations, such as soliton logic gates, are discussed.

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I. INTRODUCTION

The equation governing the propagation of a pulse in a monomode, polarization-preserving, nonlinear optical fiber in the anomalous group-velocity dispersion regime is the nonlinear Schrödinger (NLS) equation, which can be written in nondimensional form as [1,2]

$$i\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}+|u|^2u=0\ , \tag{1}$$

where u is the complex-valued envelope of the pulse. Here t represents physically the normalized spatial variable along the length of the fiber, and x is the normalized reduced time (i.e., shifted to be in a frame of reference which moves with the group velocity of a pulse). We write x and t in the NLS equation (1) in the standard mathematical, rather than optical, notational convention [3,4], but we will discuss the implications of the results for optical applications. The NLS equation has the well-known soliton solution

$$u=\eta\,\mathrm{sech}\,\eta(x-x_0-Vt)e^{(1/2)i(\eta^2-V^2)t+iVx+i\theta}\ , \tag{2}$$

where η, x_0, V, and θ are constants. (If the reference frame has been chosen to be centered at the soliton, however, then $x_0=0$ and $V=0$.) There is currently substantial interest in optical solitons due to the potentially large increase in transmission speed, or bit rate, which is likely to be obtained by their use. Solitons are ideal candidates as optical bits since they are relatively robust to perturbations.

In theory, Eq. (1) can be solved exactly using the inverse scattering transform [3], which shows that a general initial condition evolves into a fixed number of solitons plus decaying dispersive radiation. In practice, however, while the final steady solitons can be determined fairly easily, the evolution of the initial condition to these solitons is very difficult to determine using the inverse scattering transform. This is because this transient evolution is driven in large part by interactions between the emerging solitons and the dispersive radiation, and the dispersive radiation is very difficult to determine from the integral equation which is part of inverse scattering.

As an alternative, here we derive approximate equations which describe this transient evolution via the Lagrangian formulation of the NLS equation. The idea is to employ a trial function which consists of a solitonlike pulse with variable parameters plus a term which represents linear dispersive radiation. When used in the NLS Lagrangian and combined with the solution of the linear part of the NLS equation, this trial function yields ordinary differential equations (ODE's) for the pulse parameters and the amount of dispersive radiation which approximate the transient evolution. As expected, in the approximate ODE's the radiation acts as a damping, which causes the pulse to decay to a steady soliton. The solutions of the approximate ODE's also agree very well quantitatively when compared with numerical solutions of the NLS equation.

Only the NLS equation is considered here, but a number of specific applications connected with the NLS equation are expected to benefit from an improved method for determining the interaction between solitons and dispersive radiation, and these provide added motivation for the present work. These applications involve problems governed by perturbed or coupled NLS equations. The behavior of solutions in such cases has typically been determined in the past via soliton perturbation theory, which also yields equations for the evolution of the soliton parameters. A straightforward application of soliton perturbation theory [5], however, does not include interaction effects between the soliton and the dispersive radiation.

A variational trial function which partially includes the effects of dispersive radiation has already been given [6]. This variational method will be shown here to include a local interaction between the pulse and the dispersive radiation, but it makes no provision for the propagation of radiation away from the vicinity of the pulse. Additional motivation for the present work is to develop a better understanding as to why this variational trial function works as well as it does, while at the same time improving upon it. This approach has been used to study a number of applications such as pulse propagation in birefringent optical fibers [7–11], and has been shown to provide a more accurate description of the solution behavior than other methods. Recently it has been reported that the method also leads to an efficient model of soliton dragging logic gates (high-speed optical switches which employ solitons) [12]; in particular, this variational method proved to be the most accurate among the several approximate methods tested.

While including this local interaction between a soliton and dispersive radiation gives an improvement over standard soliton perturbation theory, there are still substantial differences between such approximate solutions and numerical solutions. These differences are clearly due to the propagation of dispersive radiation away from the vicinity of the soliton, which causes a permanent change in it. For additional improvements to the variational approximations, it seems necessary that this effect be included in some manner similar to the way presented here.

II. APPROXIMATE EVOLUTION EQUATIONS

In what follows, the particular initial condition

$$u(x,0)=a\,\mathrm{sech}\frac{x}{\beta},\quad -\infty<x<\infty \tag{3}$$

will be considered as a simple, specific example, since the calculations are relatively straightforward and, furthermore, since for this initial condition the final soliton state of the NLS equation is known [13]. In addition, it is possible to rescale x, t, and u in the NLS equation so that $\beta=1$ in the above, so that $\beta=1$ can be assumed without any loss of generality.

The Lagrangian density for the NLS equation (1) is

$$L=i(u^*u_t-uu_t^*)-|u_x|^2+|u|^4\ , \tag{4}$$

where $*$ denotes complex conjugation, and where u and u^* are treated as separate variables when variations are taken [6]. Key to what follows is the choice of trial function for u to substitute into this Lagrangian. For example, if one bases the trial function upon the soliton solution (2) with $x_0=0$ and $V=0$, i.e.,

$$u=\eta\,\mathrm{sech}\eta x e^{i\theta}\ , \tag{5}$$

where $\eta=\eta(t)$ and $\theta'(t)=\eta^2/2$, then one essentially obtains the result of soliton perturbation theory [8,14], namely $d\eta/dt=0$. The soliton perturbation theory result follows since the conservation laws of the NLS equation are intimately connected to the Lagrangian [4], and it is well known that the conservation laws can be used to develop perturbation theory for solitons [5,15].

A somewhat better trial function is to allow the amplitude and width in (5) to vary independently [7], but better still is to also include a quadratic phase variation across the pulse's profile (called a chirp since the local wave number varies linearly across the profile) [6,7]:

$$u=\eta\,\mathrm{sech}\frac{x}{w}e^{i\theta+ibx^2/2w}\ . \tag{6}$$

When substituted into the Lagrangian, and when variations are taken with respect to the variables η, w, θ, and chirp b, one obtains the equations [6,7]

$$\frac{d}{dt}(\eta^2 w)=0\ , \tag{7a}$$

$$\frac{dw}{dt}=b\ , \tag{7b}$$

$$\frac{db}{dt}=\frac{4}{\pi^2 w^3}(1-\eta^2 w^2)\ , \tag{7c}$$

$$\frac{d\theta}{dt}=\tfrac{5}{6}\eta^2-\frac{1}{3w^3}\ . \tag{7d}$$

Interestingly enough, these equations can also be obtained directly from the conservation laws for the NLS equation (see the Appendix). A comparison of the solution of this system of equations (the exact solution is also given in the Appendix) with the numerical solution of the NLS equation is given in Fig. 1. These equations simulate much of the behavior present in the solution of the NLS equation, but are "unable to account for the damping of the amplitude oscillations" [6].

In order to improve upon the trial function (6) used above, it is necessary to understand the role played by the chirp b. One can investigate the connection between the chirp and the dispersive radiation by linearizing about the soliton using $u=u_0+u_1$, where u_0 is the exact soli-

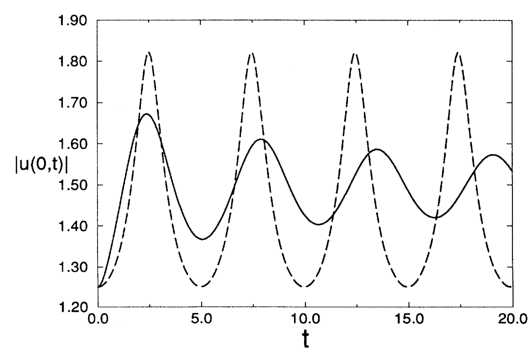

FIG. 1. Comparison between solution of equations (7) with the numerical solution of the NLS equation (1). The magnitude of u at $x=0$ for the initial condition $u(x,0)=1.25\,\mathrm{sech}x$ is plotted as a function of time. Numerical solution of NLS equation: ——; solution of approximate equations: — — —.

ton $\kappa\,\mathrm{sech}\kappa x$ and $|u_1| \ll |u_0|$. In this linear limit, the exact solution for u_1 is given by

$$u_1 = -\frac{\partial^2 f}{\partial x^2} + 2\kappa \tanh\kappa x \frac{\partial f}{\partial x} - \kappa^2 \tanh^2 \kappa x f + \kappa^2 \mathrm{sech}^2 \kappa x \exp(2i\theta) f^* , \tag{8}$$

where $\theta = \kappa^2 t/2$, and $f(x,t)$ is a solution of the linear Schrödinger equation

$$\frac{\partial f}{\partial t} = \frac{i}{2} \frac{\partial^2 f}{\partial x^2} , \tag{9}$$

with an initial condition which depends in a complicated way upon $u(x,0)$ [15]. A general characteristic of solutions of the linear Schrödinger equation (9), however, is that variations in the solution are rapidly smoothed out by the dispersive derivative $\partial^2/\partial x^2$. Thus, apart from a short initial period, $f(x,t)$ is relatively flat; in particular, very quickly $\partial f/\partial x$ and $\partial^2 f/\partial x^2 \to 0$ in the vicinity of the soliton. This shelf is clearly seen near the soliton in the numerical solution of the NLS equation shown in Fig. 2. Note that when f is approximately constant, (8) simplifies considerably, becoming

$$u_1 \approx h(1 - \mathrm{sech}^2 \kappa x) - h^* \exp(2i\theta) \mathrm{sech}^2 \kappa x , \tag{10}$$

where $h = -\kappa^2 f$. Thus we see that the solution consists of a soliton (which has a constantly increasing phase θ) plus a relatively flat and constant shelf. The oscillation in t of the peak soliton magnitude is therefore merely due to this shelf being alternately added in and out of phase with the soliton (since the soliton's phase θ is continuously increasing).

With this approximation, the solution for u becomes

$$u \approx \exp(i\theta)[\kappa\,\mathrm{sech}\kappa x + |h|(1 - 2\,\mathrm{sech}^2 \kappa x)\cos\varphi - i|h|\sin\varphi] , \tag{11}$$

where $\varphi = \theta - \arg h$. By comparing this approximate solution with the trial function (6), an interpretation of the chirp b can be obtained. When the pulse is near a soliton, we have that $\eta = \kappa + \delta\eta$ and $1/w = \kappa - \kappa^2 \delta w$. When $\delta\eta$, δw, and b are small, (6) becomes

$$u \approx \kappa\,\mathrm{sech}\kappa x + \delta\eta\,\mathrm{sech}\kappa x + \kappa^2 \delta w \kappa x\,\mathrm{sech}\kappa x \tanh\kappa x + \frac{i}{2} b \kappa^2 x^2 \mathrm{sech}\kappa x . \tag{12}$$

It has already been mentioned that Eqs. (7) can be obtained by using either the Lagrangian (4) or the conservation laws listed in the Appendix. Therefore, we can use the conservation laws to investigate the similarities between the two approximate expressions (11) and (12).

For example, if we examine the perturbation to the mass conservation law (A3a), from (11) we obtain to first order $\int \rho\, dx = 2\kappa$. From (12), however, we obtain $\int \rho\, dx = 2\kappa + 2\delta\eta + \kappa^2 \delta w$. Thus mass conservation implies to first order that $\kappa^2 \delta w = -2\delta\eta$. Note that in (11) the shelf, represented by h, does not contribute to first order to the mass perturbation. This is expected in general [15]. Similarly, if we compare the perturbations to the second moment of mass in each case, $\int x^2 \rho\, dx$, we obtain $\delta\eta = -(6/\pi)|h|\cos\varphi$, and if we compare the perturbations to the moment $\int xJ\, dx$ in each case we obtain $b = -(6/\pi)|h|\sin\varphi$.

From these results we see that the chirp b, as well as the variations in the amplitude η and width w, can all be interpreted as arising due to interactions between the soliton and the shelf h. In particular, the variations in η and w (which are linked in such a way as to keep the mass constant to first order) come from the interaction with the part of the soliton which is in phase with the shelf, while the variations in b come from the interaction with the part of the soliton which is out of phase with it. An important difference between the two approximations, however, is that the actual perturbations to the soliton presented by (11) are not localized, even though this might be expected from the trial function (6).

Another way to interpret the relationship between $\delta\eta$ and h is to compare the two approximations at $x = 0$. Equations (11) and (12) then give

$$\delta\eta = -|h|\cos\varphi . \tag{13}$$

Since we will be comparing the magnitude of the approximate and numerical solutions at $x = 0$ as a function of t, this is more appropriate. The appropriate alternative relationship for the equivalent of b will be determined shortly.

Based upon the above analysis, to determine approximate equations for the evolution of the initial condition (3), we will assume a trial function that is a combination of the above two approximate forms, namely

$$u = \left[\eta\,\mathrm{sech}\frac{x}{w} + ig\right] \exp i\theta . \tag{14}$$

Here η, w, θ, and g are functions of t. The first term represents a varying solitonlike pulse and allows the initial condition (3) to evolve smoothly to a soliton solution of the NLS equation. It also includes the variations in η and w due to the in-phase interaction with the shelf. The second term g represents the out-of-phase interaction between the soliton and the shelf. This term is assumed not to depend upon x in the vicinity of the soliton by reason

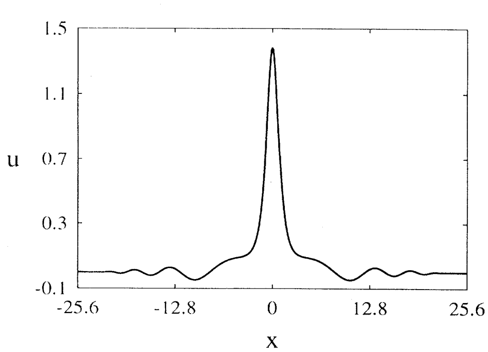

FIG. 2. Numerical solution of the NLS equation (1) for initial condition $u(x,0) = 1.25\,\mathrm{sech}x$, showing the formation of the shelf in the vicinity of the soliton.

of the approximation (10). It cannot be independent of x over an infinite range, however, as them it would contain an infinite amount of mass. We therefore assume that g contributes to the Lagrangian only in a region of length l centered about the pulse position. Since numerical solutions show that the radiation has much smaller amplitude than the pulse, it will also be assumed that $|g| \ll \eta$. The form of the radiation away from the vicinity of the pulse will be considered in Sec. III.

The approximate equations governing the evolution of the pulse are obtained from variations of the Lagrangian

$$\mathcal{L}=\int_{-\infty}^{\infty} L\,dx \ . \tag{15}$$

Substituting the form (14) for u into the Lagrangian (15), we obtain

$$\mathcal{L}=\pi g(w\eta'+\eta w')-\pi\eta w g'-2\eta^2 w\theta'-lg^2\theta' -\frac{1}{3}\frac{\eta^2}{w}+\tfrac{2}{3}\eta^4 w+2\eta^2 w g^2 \ , \tag{16}$$

plus higher order terms in g. Of the two quadratic terms in g in the Lagrangian $\mathcal{L}$, only the term proportional to l is important; the results are essentially independent of the presence or absence of the other term. The term proportional to l arises from the integration of $u^*u_t-uu_t^*$, and gives the amount of mass in the radiation in the vicinity of the pulse, as will be shown below. Only this quadratic term will be kept in what follows.

With this modification, taking variations of the Lagrangian (16) gives the equations

$$\delta\eta:\quad -\pi w g'-2\eta w\theta'-\frac{1}{3}\frac{\eta}{w}+\tfrac{4}{3}\eta^3 w=0 \ , \tag{17a}$$

$$\delta w:\quad -2\pi\eta g'-2\eta^2\theta'+\frac{1}{3}\frac{\eta^2}{w^2}+\tfrac{2}{3}\eta^4=0 \ , \tag{17b}$$

$$\delta\theta:\quad \frac{d}{dt}(2\eta^2 w+lg^2)=0 \ , \tag{17c}$$

$$\delta g:\quad \pi(\eta w)'-lg\theta'=0 \ . \tag{17d}$$

Note that the variation with respect to θ, Eq. (17c), merely expresses conservation of mass, (A3a). The term proportional to g^2 represents the amount of mass in the radiation near the pulse, and is important for driving the interaction between the dispersive radiation and the pulse. The quadratic term in g in (16) which was neglected does not lead to any terms in the mass conservation equation, and so is not needed.

After some manipulation, Eqs. (17) can be written in the form

$$\frac{d}{dt}(\eta w)=\frac{l}{\pi}g(\eta^2-\tfrac{1}{2}w^{-2}) \ , \tag{18a}$$

$$\frac{dg}{dt}=-\frac{2}{3\pi}\eta(\eta^2-w^{-2}) \ , \tag{18b}$$

$$\frac{d\theta}{dt}=\eta^2-\tfrac{1}{2}w^{-2} \ , \tag{18c}$$

$$\frac{d}{dt}\left[\frac{\eta^2}{w}-2\eta^4 w\right]=0 \ . \tag{18d}$$

The mass conservation equation (17c) follows from the first two of these equations, while the last equation can be most easily obtained from the exact expression for energy conservation for the NLS equation (A3c). The system of equations (18) is not yet complete for two reasons, however: the length l has not been specified, and no contribution from the radiation propagating away from the vicinity of the pulse has been included. The lack of radiation loss means that the solutions of these equations oscillate and do not approach a steady state.

The system of equations (18) has a fixed point at $\eta=w^{-1}\equiv\hat{\kappa}$, which is a soliton solution of the NLS equation. The length parameter l is determined by the requirement that the frequency of the oscillations of the linearized solution about this critical point matches the soliton oscillation frequency $\hat{\kappa}^2/2$, as shown in (11). This requirement gives

$$l=\frac{3\pi^2}{8\hat{\kappa}} \ . \tag{19}$$

Other choices for l were tried, such as $l=3\pi^2 w/8$, but as long as the choice taken had the same value at the fixed point there were no substantial differences in the dynamics. The fixed point, of course, can be obtained from the energy conservation equation (18d),

$$\hat{\kappa}=\left[2\eta^4 w-\frac{\eta^2}{w}\right]^{1/3} \ . \tag{20}$$

In Sec. III, we shall show how Eqs. (18) can be extended to include the radiation loss.

At this point some comparisons can be made between the results from the Anderson equations (7), the present approximate equations (18), and the inverse scattering solution of the NLS equation. The final steady soliton amplitude as given by the present approximate equations, (20), is determined by energy conservation. For the Anderson equations, however, mass conservation (17c) determines the steady-state amplitude; the result is

$$\kappa=a^2 \ , \tag{21}$$

where a is the amplitude of an initial pulse of unit width ($\beta=1$). In addition, from the inverse scattering solution of the NLS equation it was found [13] that the exact final steady-state amplitude for the initial condition (3) (with $\beta=1$) is given by

$$\kappa=2a-1 \ . \tag{22}$$

Figure 3 shows a comparison between the three final amplitudes. It can be seen that the agreement between the inverse scattering result (22) and the present result (20) is very good for initial amplitudes above about 0.9, but, as the initial amplitude decreases below 0.9, these two expressions start to diverge. Indeed, the amplitude given by (20) becomes zero when $a=1/\sqrt{2}$, while the final amplitude given by inverse scattering (22) becomes zero when $a=0.5$. This disagreement indicates that for initial amplitudes below about 0.9, the energy loss associated with the dispersive radiation becomes significant. On the other hand, it can be seen that the amplitudes given by the

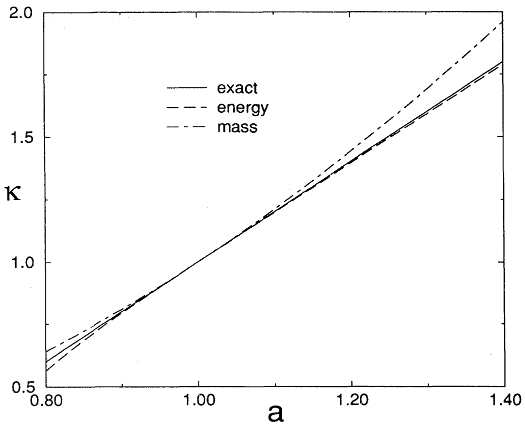

FIG. 3. Final soliton amplitude as a function of initial amplitude. Inverse scattering solution of NLS equation: ——; solution of approximate equations: — — —; solution of chirp equations: · · · .

Anderson equations (7) are in good agreement with those from inverse scattering only for initial amplitudes around 1. For $a=1$, of course, the initial condition corresponds to an exact soliton and so no pulse evolution occurs.

III. RADIATION LOSS

While the comparison in Sec. II shows that the fixed point of the approximate equations (18) is in good agreement with the exact solution of the NLS equation, the solution of these equations does not decay onto the fixed point until the effect of dispersive radiation has been added. Since the amplitude of the radiation is small, away from the pulse the nonlinear term in the NLS equation (1) is negligible. Therefore the equation governing the radiation propagating away from the pulse is

$$i\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}=0\ . \tag{23}$$

The conservation of mass for this linearized equation is the same as for the NLS equation (A2a). Integrating the differential form of the mass conservation equation from $x=l/2$ to $x=\infty$ gives the mass radiated to the right away from the vicinity of the pulse,

$$\frac{d}{dt}\int_{l/2}^{\infty}|u|^2dx=\mathrm{Im}(u^*u_x)|_{x=l/2}\ , \tag{24}$$

which by symmetry is the same as the mass radiated to the left. By Laplace transforming equation (23), it may be shown that

$$u_x(l/2,t)=-\sqrt{2}e^{-\pi i/4}\frac{d}{dt}\int_0^t\frac{u(l/2,\tau)}{\sqrt{\pi(t-\tau)}}d\tau\ . \tag{25}$$

Upon substitution into (24), this then gives

$$\frac{d}{dt}\int_{l/2}^{\infty}|u|^2dx=-\sqrt{2}\mathrm{Im}\left[e^{-\pi i/4}u^*(l/2,t)\times\frac{d}{dt}\int_0^t\frac{u(l/2,\tau)}{\sqrt{\pi(t-\tau)}}d\tau\right]\ . \tag{26}$$

The mass radiated into the two regions $x>l/2$ and $x<-l/2$ must be lost from the mass contained in the solution in the vicinity of the soliton. Hence, combining the mass conservation equation in the neighborhood of the pulse (17c) and twice the result of expression (26), we obtain a modified equation for total mass conservation,

$$\frac{d}{dt}\left[2\eta^2w+\frac{3\pi^2}{8\hat{\kappa}}g^2\right]=2\sqrt{2}\mathrm{Im}\left[e^{-\pi i/4}u^*(l/2,t)\times\frac{d}{dt}\int_0^t\frac{u(l/2,\tau)}{\sqrt{\pi(t-\tau)}}d\tau\right]\ . \tag{27}$$

In this equation, however, the solution at the edge of the shelf, $u(l/2,t)$, has not yet been identified. First of all, since the shelf is small in magnitude and relatively flat, from (23) its phase is expected to be approximately constant (i.e., slowly varying) after (possibly) an initial transient. The numerical solution of the NLS equation also bears this out. [In addition, a more careful examination of the solution of Eqs. (18) which compares the angle associated with the ηw and g oscillation and the solution for θ also leads to this conclusion.] When this phase is constant, however, Eq. (27) is independent of the phase of u and therefore $u(l/2,t)$ and $u^*(l/2,t)$ can both be replaced by $|u(l/2,t)|=|h|\equiv r$.

In addition, if one examines the oscillation produced by Eqs. (18) near the critical point using

$$\eta=\hat{\kappa}+\eta_1 \quad\text{and}\quad w=\frac{1}{\hat{\kappa}}+w_1\ , \tag{28}$$

where η_1 and w_1 are small, then the mass conservation equation (17c) becomes

$$\frac{d}{dt}\left[2\eta^2w+\frac{3\pi^2}{8\hat{\kappa}}g^2\right]\approx\frac{8}{3\hat{\kappa}}\frac{d}{dt}\left[\eta_1^2+\frac{9\pi^2}{64}g^2\right]\ . \tag{29}$$

From (13), η_1 can be identified as $-|h|\cos\varphi\equiv -r\cos\varphi$, and by comparing the imaginary parts of Eqs. (14) and (11) it is reasonable to identify g as $-|h|\sin\varphi$. This is not completely consistent with the oscillations described by Eqs. (18), and also implicitly in (29) above. This is because the integrals arising in the variational formulation produce slightly different constants than what one obtains when one merely compares the approximate solutions at $x=0$. This slight difference in the constants is similar to what happens with other variational approximations [6]. Therefore, consistent with (29), we will instead identify $3\pi g/8$ as $-|h|\sin\varphi\equiv -r\sin\varphi$. This allows us to relate the height of the shelf, r, to the deviation of mass from its value at the critical point. Equation (29) therefore gives

$$r^2\approx\eta_1^2+\frac{9\pi^2}{64}g^2\ , \tag{30}$$

near the critical point, since the height of the shelf, r, drives the rate of mass loss from the pulse, (30) relates the rate of mass loss to the total amount of mass present, when the pulse is close to a soliton. It will be assumed that this relationship between rate of mass loss and amount of mass present holds in general, so that we will take

$$r^2=\frac{3\hat{\kappa}}{8}\left[2\eta^2 w-2\hat{\kappa}+\frac{3\pi^2}{8\hat{\kappa}}g^2\right] . \tag{31}$$

The conservation of mass equation (27) then becomes

$$\frac{d}{dt}\left[2\eta^2 w+\frac{3\pi^2}{8\hat{\kappa}}g^2\right]=-2r\frac{d}{dt}\int_0^t\frac{r}{\sqrt{\pi(t-\tau)}}d\tau . \tag{32}$$

The next step is to couple the required rate of mass loss from (32) to the equations for ηw and g, (18). The mass loss can be added to either equation, but adding it to the equation for g is preferred. Recall that the mass loss is driven by the shelf, which has an approximate height r. Equation (31) gives a nonzero result for the initial height of the shelf, however, which cannot be correct since in reality the shelf takes a finite time to form. As a result, using (31) and (32) as they stand overestimates the initial mass loss rate somewhat. By moving the mass loss to the equation for g, however, this discrepancy is automatically corrected, since g is zero initially. Therefore, we merely add a term proportional to $-g$ in Eq. (18b) and adjust the multiplying function so the average mass loss rate matches that obtained from (32) near the critical point. The result is

$$\frac{dg}{dt}=\frac{2}{3\pi}\frac{\eta}{w^2}(1-\eta^2 w^2)-2\alpha g , \tag{33}$$

where

$$\alpha=\frac{3\hat{\kappa}}{8}\frac{1}{r}\frac{d}{dt}\int_0^t\frac{r}{\sqrt{\pi(t-\tau)}}d\tau . \tag{34}$$

The equation for ηw, (18a), remains the same.

Finally, an additional approximation can be made in this particular case since it is an initial-value problem. First of all, the integrodifferential equations (31) and (32) (note that $\hat{\kappa}$ is constant) for the total amount of mass loss can be solved using Laplace transforms. The result is

$$r=\frac{r(0)}{\sqrt{\pi}}e^{\gamma t}\int_{\gamma t}^{\infty}\zeta^{-1/2}e^{-\zeta}d\zeta , \tag{35}$$

where the constant γ is

$$\gamma=\frac{9\hat{\kappa}^2}{64} . \tag{36}$$

Now, as $t\to 0$, this solution has the same asymptotic expansion to second order as the solution of

$$\frac{dr}{dt}=-\sqrt{\gamma}\frac{r^2}{r(0)\sqrt{\pi t}} . \tag{37}$$

In addition, the solution of this equation has the same asymptotic behavior in the limit $t\to\infty$ as (35). Hence, comparing (35) and (37), we can make the approximation

$$\frac{d}{dt}\int_0^t\frac{r}{\sqrt{\pi(t-\tau)}}d\tau\approx\frac{r^2}{r(0)\sqrt{\pi t}} . \tag{38}$$

Using this expression to replace the loss in (27), we have that the approximate mass conservation equation including loss due to dispersive radiation is

$$\frac{d}{dt}\left[2\eta^2 w+\frac{3\pi^2}{8\hat{\kappa}}g^2\right]\approx-\frac{2r^3}{r(0)\sqrt{\pi t}} , \tag{39}$$

on using (31) and (32). This particular approximation is not appropriate for perturbed NLS problems, of course, when dispersive radiation is likely to be shed continuously.

The full system of equations describing the evolution of the initial pulse (3), including radiation loss, is then

$$\frac{dg}{dt}=\frac{2}{3\pi}\frac{\eta}{w^2}(1-\eta^2 w^2)-2\alpha g , \tag{40a}$$

$$\frac{d}{dt}(\eta w)=\frac{3\pi}{8\hat{\kappa}}(\eta^2-\tfrac{1}{2}w^{-2})g , \tag{40b}$$

$$w=\frac{1}{\hat{\kappa}^3}\sqrt{2(\eta w)^4-(\eta w)^2} , \tag{40c}$$

$$\alpha=\frac{3\hat{\kappa}}{8}\frac{1}{r}\frac{d}{dt}\int_0^t\frac{r}{\sqrt{\pi(t-\tau)}}d\tau\approx\frac{3\hat{\kappa}}{8}\frac{1}{r}\frac{r^2}{r(0)\sqrt{\pi t}} , \tag{40d}$$

$$r^2=|h|^2=\frac{3\hat{\kappa}}{8}\left[2\eta^2 w+\frac{3\pi^2}{8\hat{\kappa}}g^2-2\hat{\kappa}\right] , \tag{40e}$$

$$\frac{d\theta}{dt}=\eta^2-\tfrac{1}{2}w^{-2} . \tag{40f}$$

IV. COMPARISON WITH NUMERICAL SOLUTIONS

The approximate evolution equations for the soliton (40) were solved numerically using a fourth-order Runge-Kutta method with the value of $\hat{\kappa}$ (20) calculated from the initial values of η and w (a and β). To reduce computation time by solving a third-, rather than a fourth-order system, Eqs. (40a), (40b), and (40f) were solved numerically for g, ηw, and θ. The values for w, and hence η, were obtained from (40c), which is a consequence of the energy conservation equation (18d), and the known value of ηw. The full numerical solutions of the NLS equation (1) were obtained by using a pseudospectral method similar to that used for the Korteweg–de Vries equation [16].

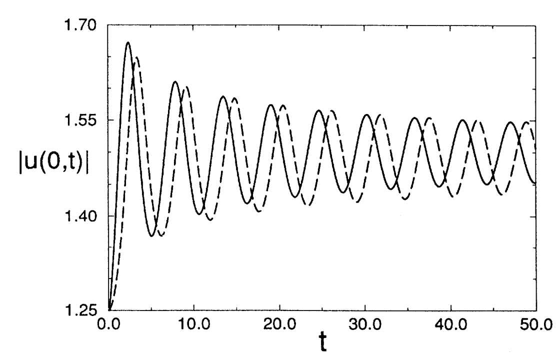

FIG. 4. Soliton amplitude η as a function of t for $a=1.25$, $\beta=1$. Numerical solution of NLS equation: ——; solution of approximate equations: — — —.

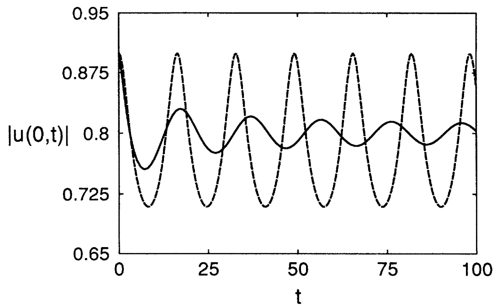

FIG. 5. Soliton amplitude η as a function of t for $a=0.9$, $\beta=1$. Numerical solution of the NLS equation: ——; solution of chirp equations: — — —.

Figure 4 shows a comparison between the amplitude $u(0,t)$ of the soliton as given by the approximate equations and the full numerical solution of the NLS equation (1). The initial conditions used were $a=1.25$ (and $\beta=1$). It can be seen that the comparison is very good, with the period of the oscillations of the solution of the approximate equations in good agreement with the actual period, but with a relatively small phase shift between these oscillations. This phase shift appears to result from assuming the phase of the shelf is approximately constant, which is not precisely true during its early stages of formation. Since in many applications an accurate determination of the phase of the solution is not important, this phase shift is not critical. In addition, it should be possible to solve the integrodifferential equation (26) numerically to more accurately recover the phase.

As was discussed at the end of Sec. II, as the initial amplitude a decreases below 1.0, the difference between the final amplitude as given by the approximate equations and the inverse scattering solution of the NLS equation increases. This can be seen in Figs. 5 and 6, where the solutions of the two sets of approximate equations are compared with the numerical solution of the NLS equation for an initial amplitude $a=0.9$ (with $\beta=1$). The solution of the approximate equations including the chirp term (7) is shown in Fig. 5. In addition to the obvious difference due to the lack of damping of the oscillations in the chirp equations, the period of the oscillations obtained from the chirp equations is significantly shorter than the actual period. Figure 6 shows the same comparison for the present approximation, (40). It can be seen from the figure that the period of the oscillations arising in the solution of these approximate equations is only slightly longer than the actual period, and that with the inclusion of damping the approximate solution becomes significantly more accurate.

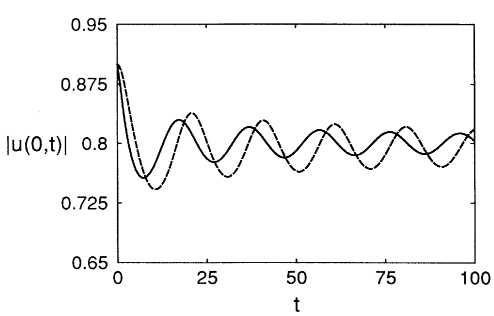

FIG. 6. Soliton amplitude η as a function of t for $a=0.9$, $\beta=1$. Numerical solution of NLS equation: ——; solution of approximate equations: — — —.

V. CONCLUSIONS

We have examined the time dependent behavior of solutions of the nonlinear Schrödinger equation. While in principle the exact inverse scattering solution of the NLS equation can be used to determine such transient behavior, in practice it is difficult to do so. As an alternative, a Lagrangian approach has been used here to derive approximate equations which describe the transient soliton evolution. These approximate equations include the mass loss due to dispersive radiation, and were found to give solutions in very good agreement with full numerical solutions of the NLS equation.

The method developed here has the further advantage that it can be applied to problems for which there are no inverse scattering solutions. An example of current interest is the system of coupled NLS equations,

$$i\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}+(|u|^2+A|v|^2)u=0 ,$$

$$i\frac{\partial v}{\partial t}+\frac{1}{2}\frac{\partial^2 v}{\partial x^2}+(|v|^2+A|u|^2)v=0 ,$$

which models the propagation of pulses in birefringent optical fibers [7,17]. This coupled system of equations possesses an inverse scattering solution only for the cases $A=0$ and 1, while the values of A of physical interest in optical fiber problems are $A=2/3$ (linear polarization) and 2 (circular polarization). Approximate equations for pulse evolution for these equations have been determined [7] using chirp method [6], and good agreement has been found with full numerical solutions of the equations for short times. For longer times, the effect of dispersive radiation becomes important, and the difference between the two solutions increases. The method of the present work can be extended to this coupled system, so that the effect of dispersive radiation on the pulse evolution is included. This is expected to improve the agreement between the approximate and full numerical solutions, and will be the subject of future work.

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APPENDIX A: CONSERVATION LAWS

We first define the quantities

$$\rho=|u|^2 , \qquad \text{(A1a)}$$

$$J=\frac{i}{2}(uu_x^*-u^*u_x)\ , \tag{A1b}$$

$$E=|u_x|^2-|u|^4\ , \tag{A1c}$$

which are generally referred to as the mass density, mass flux (or momentum) density, and energy density [5]. (Note, however, that in the optical context mass should really be photon number.) Directly from the NLS equation, the following conservation laws can be derived:

$$\frac{\partial\rho}{\partial t}+\frac{\partial J}{\partial x}=0\ , \tag{A2a}$$

$$\frac{\partial J}{\partial t}+\frac{\partial}{\partial x}[E+\tfrac{1}{2}\rho^2-\tfrac{1}{4}\rho_{xx}]=0\ , \tag{A2b}$$

$$\frac{\partial E}{\partial t}+\frac{\partial}{\partial x}\left[\frac{i}{2}(u_xu_{xx}^*-u_x^*u_{xx})-2\rho J\right]=0\ . \tag{A2c}$$

By integrating with respect to x, the conservation laws are obtained in their standard forms, namely

$$\frac{d}{dt}\int_{-\infty}^{\infty}\rho\,dx=0\ , \tag{A3a}$$

$$\frac{d}{dt}\int_{-\infty}^{\infty}J\,dx=0\ , \tag{A3b}$$

$$\frac{d}{dt}\int_{-\infty}^{\infty}E\,dx=0\ . \tag{A3c}$$

These conservation laws are directly associated with invariances of the NLS Lagrangian (the first with invariance with respect to phase changes, the second with translations in x, and the third with translations in t). In addition, however, it is possible to obtain equations for the various moments of these densities,

$$\frac{d}{dt}\int_{-\infty}^{\infty}x\rho\,dx=\int_{-\infty}^{\infty}J\,dx\ , \tag{A4a}$$

$$\frac{d}{dt}\int_{-\infty}^{\infty}x^2\rho\,dx=2\int_{-\infty}^{\infty}xJ\,dx\ , \tag{A4b}$$

$$\frac{d}{dt}\int_{-\infty}^{\infty}xJ\,dx=\int_{-\infty}^{\infty}[E+\tfrac{1}{2}\rho^2]dx\ . \tag{A4c}$$

The first of these moment equations can be associated with the Galilean invariance of the NLS Lagrangian, but it is not clear which invariances might yield the other two equations. These moment equations are obviously not closed in their present form, but they do close if one assumes a particular form for the solution, such as Eq. (6). With this assumption, one immediately obtains the first three of Eqs. (7). (Note that half of the six integrals vanish since the integrands are odd.) The moment equations also give a motivation for including the chirp b in the trial function (6): a term in the phase proportional to x^2 is a reasonable choice for making the moment $\int xJ\,dx$ nonzero. From the above equations it is clear that this particular moment is intimately connected with variations in the width of the soliton through $\int x^2\rho\,dx$.

One cannot obtain an equation for the phase θ from any of these conservation laws (including the higher-order conservation laws [3]), as they are all independent of the phase. There is a fifth invariance of the NLS equation, however: scale invariance. This is not an invariance of the Lagrangian, though, since it must be rescaled as well. Nevertheless, Noether's theorem [4] can be modified to give something akin to a conservation law in this case:

$$\frac{d}{dt}\int_{-\infty}^{\infty}xJ\,dx+\int_{-\infty}^{\infty}\frac{i}{2}(u^*u_t-uu_t^*)dx-\tfrac{3}{2}\int_{-\infty}^{\infty}E\,dx=0\ . \tag{A5}$$

This modified conservation law can be used to derive the equation for the time evolution of the NLS phase θ. As such, it should perhaps be thought of as a replacement for the last (sixth) moment equation.

The solution of Eqs. (7) for η, w, and b can be found explicitly. First, η is eliminated in favor of w using conservation of mass

$$\eta^2w=a^2\beta\equiv\hat{\kappa}\ , \tag{A6}$$

where a and β are the initial values of η and w. Since $\eta w=1$ is the steady state of the system, $\hat{\kappa}$ is an approximation to the steady-state soliton amplitude. The solution of the equations for w and b can then be written

$$\frac{1}{w}=\hat{\kappa}(1-e\cos\tau)\ , \tag{A7}$$

$$b=-\frac{2}{\pi}\hat{\kappa}e\sin\tau\ , \tag{A8}$$

where τ is given implicitly as a function of t by

$$\frac{2}{\pi}\hat{\kappa}^2t=\frac{1}{(1-e^2)}\left\{\frac{2}{\sqrt{1-e^2}}\tan^{-1}\left[\left[\frac{1+e}{1-e}\right]^{1/2}\tan\frac{\tau}{2}\right]+\frac{e\sin\tau}{1-e\cos\tau}\right\}\ , \tag{A9}$$

and where $e=1-1/a^2\beta^2$. The approximate oscillation period is then directly found to be

$$T=\frac{\pi^2}{\hat{\kappa}^2(1-e^2)^{3/2}}\ . \tag{A10}$$

For small amplitude oscillations, $e\ll1$, and we obtain $T\approx\pi^2/\hat{\kappa}^2$. The correct answer, from (11), should be $4\pi/\kappa^2$, where κ is the true steady-state soliton amplitude.

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